

# ISOMORPHISMS BETWEEN ALGEBRAS OF SEMICLASSICAL PSEUDODIFFERENTIAL OPERATORS

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**ABSTRACT.** Following the work of Duistermaat-Singer [DS] on isomorphisms of algebras of global pseudodifferential operators, we classify isomorphisms of algebras of microlocally defined semiclassical pseudodifferential operators. Specifically, we show that any such isomorphism is given by conjugation by  $A = BF$ , where  $B$  is a microlocally elliptic semiclassical pseudodifferential operator, and  $F$  is a microlocal  $h$ -FIO associated to the graph of a local symplectic transformation.

## 1. INTRODUCTION

In the study of pseudodifferential operators on manifolds, there are two important regimes to keep in mind. The first is a global study of pseudodifferential operators defined using the local Fourier transform on the cotangent bundle. If  $X$  is a compact smooth manifold and  $T^*X$  is the cotangent bundle with the local coordinates  $\rho = (x, \xi)$ , we study pseudodifferential operators with principal symbol homogeneous at infinity in the  $\xi$  variables. Let  $Y$  be another compact smooth manifold of the same dimension as  $X$ , and suppose there is an algebra isomorphism from the algebra of all pseudodifferential operators on  $X$  (filtered by order) to the same algebra on  $Y$ , and suppose that isomorphism preserves the order of the operator. Then Duistermaat-Singer [DS] have shown that this isomorphism is necessarily given by conjugation by an elliptic Fourier Integral Operator (FIO).

The other setting is the semiclassical or “small- $h$ ” regime. One can study globally defined semiclassical pseudodifferential operators, but many times it is meaningful to study operators which are microlocally defined in some small set (see §2 for definitions). Then we think of the  $h$  parameter as being comparable to  $|\xi|^{-1}$  in the global, non-semiclassical regime. Thus the study of small  $h$  asymptotics in the microlocally defined regime should correspond to the study of high frequency asymptotics in the global regime. We therefore expect a similar result to that presented in [DS], although the techniques used in the proof will vary slightly.

Let  $X$  be a smooth manifold,  $\dim X = n \geq 2$ , and assume  $U \subset T^*X$  is an open set. Let  $Y$  be another smooth manifold,  $\dim Y = n$ , and let  $V \subset T^*Y$ . Let  $\Psi^0/\Psi^{-\infty}(U)$  denote the algebra of semiclassical pseudodifferential operators defined microlocally in  $U$  filtered by the order in  $h$ , and similarly for  $V$  (see §2 for definitions).

**Theorem 1.** *Suppose*

$$g : \Psi^0/\Psi^{-\infty}(U) \rightarrow \Psi^0/\Psi^{-\infty}(V)$$

is an order preserving algebra isomorphism. For every  $\tilde{U} \Subset U$  open and precompact, there is a symplectomorphism

$$\kappa : \tilde{U} \rightarrow \overline{\kappa(\tilde{U})}$$

and  $h_0 > 0$  such that, if  $F$  is the  $h$ -FIO associated to  $\kappa$ , for all  $0 < h < h_0$  and all  $P \in \Psi^0 / \Psi^{-\infty}(U)$  we have

$$(1.1) \quad g(P) = BFPP^{-1}B^{-1} \text{ microlocally in } \kappa(\tilde{U}) \times \kappa(\tilde{U}),$$

where  $B \in \Psi^0(V)$  is elliptic on  $\kappa(\tilde{U})$ .

To put Theorem 1 in context, we observe that every algebra homomorphism of the form (1.1) is an order preserving algebra isomorphism, according to Proposition 2.3 in §2.

Automorphisms of algebras of pseudodifferential operators have also been studied in the context of the more abstract Berezin-Toeplitz quantization in [Zel].

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## 2. PRELIMINARIES

Let  $\mathcal{C}^\infty(T^*X)$  denote the algebra of smooth,  $\mathbb{C}$ -valued functions on  $T^*X$ , and define the global symbol classes

$$\mathcal{S}^m(T^*X) = \{a \in \mathcal{C}^\infty((0, h_0]_h; \mathcal{C}^\infty(T^*X)) : |\partial^\alpha a| \leq C_\alpha h^{-m}\}.$$

We define the essential support of a symbol by complement:

$$\begin{aligned} \text{ess-supp}_h(a) &= \\ &= \mathbb{C} \setminus \{(x, \xi) \in T^*X : |\partial^\alpha a| \leq C_\alpha h^N \ \forall N \text{ and } \forall (x', \xi') \text{ near } (x, \xi)\}. \end{aligned}$$

By multiplying elements of  $\mathcal{S}^m(T^*X)$  by an appropriate cutoff in  $\mathcal{C}_c^\infty(U)$ , we may think of symbols as being microlocally defined in  $U$ , and define the class of symbols with essential support in  $U$

$$\mathcal{S}^m(U) = \{a \in \mathcal{C}^\infty((0, 1]_h; \mathcal{C}_c^\infty(U)) : |\partial^\alpha a| \leq C_\alpha h^{-m}\}.$$

We write  $\mathcal{S}^m = \mathcal{S}^m(U)$  when there is no ambiguity. We can think of elements of  $\mathcal{S}^m$  as formal power series in  $h$ :

$$a(x, \xi; h) = \sum_{j=-m}^{\infty} h^j a_j(x, \xi; h),$$

where each  $a_j$  is in  $\mathcal{C}_c^\infty(U)$  and has derivatives of all orders bounded in  $h$ .

We have the corresponding spaces of pseudodifferential operators  $\Psi^m(U)$  acting by the local formula (Weyl calculus)

$$\text{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int \int a\left(\frac{x+y}{2}, \xi; h\right) e^{i\langle x-y, \xi \rangle / h} u(y) dy d\xi.$$

For  $A = \text{Op}_h^w(a)$  and  $B = \text{Op}_h^w(b)$ ,  $a \in \mathcal{S}^m$ ,  $b \in \mathcal{S}^{m'}$  we have the composition formula (see, for example, the review in [DiSj])

$$(2.1) \quad A \circ B = \text{Op}_h^w(a \# b),$$

where

$$(2.2) \quad \mathcal{S}^{m+m'} \ni a \# b(x, \xi) := e^{\frac{i\hbar}{2} \omega(D_x, D_\xi; D_y, D_\eta)} (a(x, \xi) b(y, \eta)) \Big|_{\substack{x=y \\ \xi=\eta}},$$

with  $\omega$  the standard symplectic form. Observe  $\#$  preserves essential support in the sense that if  $\text{ess-sup}_h(a) \cap \text{ess-sup}_h(b) = \emptyset$ , then  $a \# b = \mathcal{O}(h^\infty)$ . We define the wavefront set of a pseudodifferential operator  $A = \text{Op}_h^w(a)$  as

$$\text{WF}_h(A) = \text{ess-sup}_h(a),$$

so that  $\Psi^m(U)$  is the class of pseudodifferential operators with wavefront set contained in  $U$ . We denote

$$\begin{aligned} \Psi^0(U) &:= \bigcup_{m \leq 0} \Psi^m(U) \text{ and} \\ \Psi^{-\infty}(U) &:= \bigcap_{m \in \mathbb{Z}} \Psi^m(U). \end{aligned}$$

We will need the definition of microlocal equivalence of operators. Suppose  $T : \mathcal{C}^\infty(X) \rightarrow \mathcal{C}^\infty(X)$  and that for any seminorm  $\|\cdot\|_1$  on  $\mathcal{C}^\infty(X)$  there is a second seminorm  $\|\cdot\|_2$  on  $\mathcal{C}^\infty(X)$  such that

$$\|Tu\|_1 = \mathcal{O}(h^{-M_0}) \|u\|_2$$

for some  $M_0$  fixed. Then we say  $T$  is *semiclassically tempered*. We assume for the rest of this paper that all operators satisfy this condition (see [EvZw, Chap. 10] for more on this). Let  $U, V \subset T^*X$  be open pre-compact sets. We think of operators defined microlocally near  $V \times U$  as equivalence classes of tempered operators. The equivalence relation is

$$T \sim T' \iff A(T - T')B = \mathcal{O}(h^\infty) : \mathcal{D}'(X) \rightarrow \mathcal{C}^\infty(X)$$

for any  $A, B \in \Psi_h^{0,0}(X)$  such that

$$\begin{aligned} \text{WF}_h(A) &\subset \tilde{V}, \quad \text{WF}_h(B) \subset \tilde{U}, \text{ with } \tilde{V}, \tilde{U} \text{ open and} \\ \overline{V} &\Subset \tilde{V} \Subset T^*X, \quad \overline{U} \Subset \tilde{U} \Subset T^*X. \end{aligned}$$

In the course of this paper, when we say  $P = Q$  *microlocally* near  $U \times V$ , we mean for any  $A, B$  as above,

$$APB - AQB = \mathcal{O}_{L^2 \rightarrow L^2}(h^\infty),$$

or in any other norm by the assumed pre-compactness of  $U$  and  $V$ . Similarly, we say  $B = T^{-1}$  on  $V \times V$  if  $BT = I$  microlocally near  $U \times U$  and  $TB = I$  microlocally near  $V \times U$ . Thus

$$\Psi^0 / \Psi^{-\infty}(U)$$

is the algebra of bounded semiclassical pseudodifferential operators defined microlocally in  $U$  modulo this equivalence relation. It is interesting to observe that this equivalence relation has a different meaning in the high-frequency regime. There,  $\Psi^{-\infty}(X)$  corresponds to smoothing operators, although they may not be “small” in the sense of  $h \rightarrow 0$ .

We have the principal symbol map

$$\sigma_h : \Psi^m(U) \rightarrow \mathcal{S}^m / \mathcal{S}^{m-1}(U),$$

which gives the left inverse of  $\text{Op}_h^w$  in the sense that

$$\sigma_h \circ \text{Op}_h^w : \mathcal{S}^m(U) \rightarrow \mathcal{S}^m / \mathcal{S}^{m-1}(U)$$

is the natural projection.

We will use the following well-known semiclassical version of Egorov's theorem (see [Ch1, Ch2] or [EvZw] for a proof).

**Proposition 2.1.** *Suppose  $U$  is an open neighbourhood of  $(0, 0)$  and  $\kappa : \overline{U} \rightarrow \overline{U}$  is a symplectomorphism fixing  $(0, 0)$ . Then there is a unitary operator  $F : L^2 \rightarrow L^2$  such that for all  $A = \text{Op}_h^w(a)$ ,*

$$AF = FB \text{ microlocally on } U \times U,$$

where  $B = \text{Op}_h^w(b)$  for a Weyl symbol  $b$  satisfying

$$b = \kappa^* a + \mathcal{O}(h^2).$$

$F$  is microlocally invertible in  $U \times U$  and  $F^{-1}AF = B$  microlocally in  $U \times U$ .

Observe that Proposition 2.1 implicitly identifies a neighbourhood  $U$  with its coordinate representation. To make a global statement, we use the following Lemma.

**Lemma 2.2.** *Let  $U_1, U_2 \subset \mathbb{R}^{2n}$  be open sets with  $H^1(U_j, \mathbb{C}) = \{0\}$ ,  $j = 1, 2$ . Assume  $V := U_1 \cap U_2 \neq \emptyset$  and let  $\tilde{U}$  be a neighbourhood of  $U_1 \cup U_2$ . Suppose*

$$\kappa : \tilde{U} \rightarrow \kappa(\tilde{U}) \subset \mathbb{R}^{2n}$$

*is a symplectomorphism and let  $F_j$  be the quantization of  $\kappa|_{U_j}$ ,  $j = 1, 2$ , as in Proposition 2.1. Then*

$$F_1 F_2^* = \text{id} + \mathcal{O}(h^2) \text{ microlocally near } \kappa(V) \times \kappa(V) \text{ and}$$

$$F_1^* F_2 = \text{id} + \mathcal{O}(h^2) \text{ microlocally near } V \times V$$

*as pseudodifferential operators.*

*Proof.* From [Ch2, Corollary 3.4] we can for  $0 \leq t \leq 1$  find a family of symplectomorphisms  $\kappa_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ , a Hamiltonian  $q_t$ , and linear operators  $\tilde{F}_j(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ ,  $j = 1, 2$  satisfying:

$$\kappa_0 = \text{id}, \quad \kappa_1|_{\text{neigh}(V)} = \kappa,$$

$$\frac{d}{dt} \kappa_t = (\kappa_t)_* H_{q_t},$$

and if  $Q_t = \text{Op}_h^w(q_t)$  is the quantization of  $q_t$ , the  $\tilde{F}_j$  satisfy

$$hD_t \tilde{F}_j(t) + \tilde{F}_j(t) Q(t) = 0, \quad (0 \leq t \leq 1)$$

$$\tilde{F}_j(1) = F_j,$$

and  $\tilde{F}_j(0) = \text{id} + \mathcal{O}(h^2)$  as a pseudodifferential operator. The adjoints satisfy

$$hD_t \tilde{F}_j^*(t) - Q(t) \tilde{F}_j^*(t) = 0, \quad (0 \leq t \leq 1)$$

$$\tilde{F}_j^*(1) = F_j^*,$$

and  $\tilde{F}_j^*(0) = \text{id} + \mathcal{O}(h^2)$  as a pseudodifferential operator. A calculation shows  $\tilde{F}_1 \tilde{F}_2^*$  and  $\tilde{F}_1^* \tilde{F}_2$  are constant. The conclusion of the Lemma holds at  $t = 0$ , so it holds at  $t = 1$  as well.  $\square$

The next proposition is a more global version of Proposition 2.1.

**Proposition 2.3.** *Suppose  $U \subset T^*X$  is an open set and  $\kappa : \overline{U} \rightarrow \overline{\kappa U}$  is a symplectomorphism. Then for any  $\tilde{U} \Subset U$  open and precompact, there is a linear operator  $F : L^2(X) \rightarrow L^2(X)$ , microlocally invertible in  $\tilde{U} \times \kappa(\tilde{U})$  such that for all  $A = \text{Op}_h^w(a)$ ,*

$$F^*AF = B \text{ microlocally in } \tilde{U} \times \tilde{U},$$

for  $B = \text{Op}_h^w(b)$  for a Weyl symbol  $b$  satisfying

$$b = \kappa^*a + \mathcal{O}(h^2).$$

*Proof.* The idea of the proof is to glue together operators from Proposition 2.1 with a partition of unity. Let

$$1 = \sum_j \chi_j$$

be a partition of unity of  $U$  so that  $H^1(\text{supp } \chi_j, \mathbb{C}) = \{0\}$  for each  $j$ . Let  $U_j = \text{supp } \chi_j \cap U$ ,  $V_j = \kappa(U_j)$ , and let  $F_j$  be the quantization of  $\kappa|_{U_j}$  as in Proposition 2.1. Set  $F = \sum_j F_j \chi_j^w$ , where  $\chi_j^w = \text{Op}_h^w(\chi_j)$ , so that  $F^* = \sum_k \chi_k^w F_k^*$ . We first verify:

$$\begin{aligned} F^*F &= \sum_{j,k} \chi_k^w F_k^* F_j \chi_j^w \\ &= \sum_{j,k} \chi_k^w (1 + \mathcal{O}_{j,k}(h^2)) \chi_j^w \\ &= 1 + \mathcal{O}(h^2) \end{aligned}$$

microlocally on  $\tilde{U}$  since  $\tilde{U}$  is covered by finitely many of the  $U_j$ s. Further,

$$\begin{aligned} FF^* &= \sum_{j,k} F_j \chi_j^w \chi_k^w F_k^* \\ &= \sum_{j,k} \text{Op}((\kappa^{-1})^* \chi_j + \mathcal{O}_j(h^2)) F_j F_k^* \text{Op}((\kappa^{-1})^* \chi_k + \mathcal{O}_k(h^2)) \\ &= 1 + \mathcal{O}(h^2) \end{aligned}$$

as above. Hence  $F^*$  is an approximate left and right inverse microlocally on  $\overline{\tilde{U}} \times \overline{\kappa(\tilde{U})}$  so  $F$  is microlocally invertible.

Now for each  $j$ , choose  $\tilde{\chi}_j \in \mathcal{C}_c^\infty(T^*X)$  satisfying  $\tilde{\chi}_j \equiv 1$  on  $\text{supp } \chi_j$  with support in a slightly larger set so that

$$\sum_j \tilde{\chi}_j \leq C \text{ on } \tilde{U},$$

for  $C > 0$  fixed. We calculate for  $A = \text{Op}_h^w(a)$ :

$$\begin{aligned} F^*AF &= \sum_{j,k} \chi_k^w F_k^* A F_j \chi_j^w \\ &= \sum_{j,k} \chi_k^w F_k^* F_j B_j \chi_j^w, \end{aligned}$$

where  $B_j = \text{Op}_h^w(b_j)$  for a symbol

$$b_j = (\kappa^*a + \mathcal{O}(h^2)) \tilde{\chi}_j.$$

Then from Lemma 2.2 we have

$$(2.3) \quad F^*AF = \sum_{j,k} \chi_k^w B_j (1 + \mathcal{O}_{j,k}(h^2)) \chi_j^w,$$

and since we can cover  $\widetilde{U}$  with finitely many of the  $U_j$ , the error in (2.3) is  $\mathcal{O}(h^2)$  microlocally on  $\widetilde{U}$  and the Proposition follows.  $\square$

**Remark.** This notion of quantization of symplectic transformations is a constructive version of the more general definition due to Hörmander-Melrose [Hor1, Hor2, Mel] as an integral operator with a distribution kernel supported on the Lagrangian submanifold associated to the symplectic relation (see also [Dui] and the recent semiclassical treatment in [Ale]).

Let  $Y$  be another smooth manifold of the same dimension as  $X$ , and let  $V \subset T^*Y$  be a non-empty, pre-compact, open set. We say

$$g : \Psi^0/\Psi^{-\infty}(U) \rightarrow \Psi^0/\Psi^{-\infty}(V)$$

is an order preserving algebra isomorphism (of algebras filtered by powers of  $h$ ) if

$$g(\Psi^m(U)) = \Psi^m(V), \quad g^{-1}(\Psi^m(V)) = \Psi^m(U),$$

and for every  $A, A' \in \Psi^m(U)$ ,  $B \in \Psi^{m'}(U)$ ,

$$\begin{aligned} g(A + A') &= g(A) + g(A') \bmod \Psi^{-\infty}(V), \\ g(AB) &= g(A)g(B) \bmod \Psi^{-\infty}(V). \end{aligned}$$

### 3. THE PROOF OF THEOREM 1

We break the proof of Theorem 1 into several lemmas.

**Lemma 3.1.** *The maximal ideals of  $\mathcal{S}^0/\mathcal{S}^{-1}(U) + \mathbb{C}$  are either of the form*

$$(3.1) \quad \mathcal{M}_\rho := \{p \in \mathcal{S}^0/\mathcal{S}^{-1}(U) : p(\rho) = 0, \rho \in U\},$$

or

$$(3.2) \quad \mathcal{M}_{\partial U} := \mathcal{C}_c^\infty(U) + \mathcal{O}(h^\infty)\mathcal{C}^\infty(U).$$

*Proof.* Clearly for each  $\rho \in U$ ,  $\mathcal{M}_\rho$  is a maximal ideal. Also,  $\mathcal{M}_{\partial U}$  is maximal, since any ideal  $\mathcal{M}$  satisfying

$$\mathcal{M}_{\partial U} \subsetneq \mathcal{M}$$

must contain a constant, and therefore is equal to  $\mathcal{S}^0/\mathcal{S}^{-1}(U) + \mathbb{C}$ . Suppose  $\mathcal{M}$  is another maximal ideal which is not of the form (3.1) for any  $\rho \in U$ . Then for each point  $\rho \in U$ , there is  $a_\rho \in \mathcal{M}$  such that  $a_\rho(\rho) \neq 0$ . Further, by multiplying by a (positive or negative) constant if necessary, we may assume for each  $\rho$  there is a neighbourhood  $U_\rho$  of  $\rho$  such that  $a_\rho|_{\overline{U_\rho}} \geq 1$ . Let  $a(x, \xi) \in \mathcal{M}_{\partial U}$ , and let

$$K = \text{ess-supp}_h(a) \Subset U.$$

As  $K$  is compact, we can cover it with finitely many of the  $U_\rho$ ,

$$K \subset U_{\rho_1} \cup \cdots \cup U_{\rho_m},$$

and

$$b := \sum_{j=1}^m a_{\rho_j} \in \mathcal{M}$$

satisfies  $b \geq 1$  on  $K$ . Thus  $a/b \in \mathcal{M}_{\partial U}$  implies

$$a = \left(\frac{a}{b}\right) b \in \mathcal{M}.$$

Thus  $\mathcal{M}_{\partial U} \subset \mathcal{M}$ . But  $\mathcal{M}_{\partial U}$  is maximal, so either  $\mathcal{M} = \mathcal{M}_{\partial U}$  or  $\mathcal{M} = \mathcal{S}^0/\mathcal{S}^{-1}(U) + \mathbb{C}$ .  $\square$

The following three lemmas are a semiclassical version of [DS] with a few modifications to the proofs.

**Lemma 3.2.** *Suppose  $g : \Psi^0/\Psi^{-\infty}(U) \rightarrow \Psi^0/\Psi^{-\infty}(V)$  is an order preserving algebra isomorphism. Then there exists a diffeomorphism  $\kappa : U \rightarrow V$ .*

*Proof.* We first “unitalize” our algebra of pseudodifferential operators by adding constant multiples of identity. That is, let

$$\tilde{\mathcal{S}}^m(U) = \{a \in \mathcal{C}^\infty((0, 1]_h; \mathcal{C}_c^\infty(U) + \mathbb{C}) : |\partial^\alpha a| \leq C_\alpha h^{-m}\},$$

and let  $\tilde{\Psi}^m(U) = \text{OP}\tilde{\mathcal{S}}^m(U)$ . We extend  $g$  to an isomorphism

$$\tilde{g} : \tilde{\Psi}^0/\Psi^{-\infty}(U) \rightarrow \tilde{\Psi}^0/\Psi^{-\infty}(V)$$

by defining for  $C \in \mathbb{C}$  and  $P \in \Psi^0(U)$

$$\tilde{g}(C + P) := C + g(P).$$

Observe  $\tilde{g}$  induces an algebra isomorphism

$$g_0 : \tilde{\mathcal{S}}^0/\tilde{\mathcal{S}}^{-1}(U) \rightarrow \tilde{\mathcal{S}}^0/\tilde{\mathcal{S}}^{-1}(V).$$

Since  $g_0$  takes maximal ideals to maximal ideals, we can define a map

$$\kappa : U \rightarrow V.$$

First note that since  $g_0 : \mathcal{C}_c^\infty(U) \rightarrow \mathcal{C}_c^\infty(V)$ ,

$$g_0(\mathcal{M}_{\partial U}) = \mathcal{M}_{\partial V}.$$

Then for general  $\rho \in U$ , define  $\kappa : U \rightarrow V$  by

$$g_0(\mathcal{M}_\rho) = \mathcal{M}_{\kappa(\rho)}.$$

By applying  $g_0^{-1}$ , we immediately see  $\kappa$  is bijective.

Now for  $p \in \tilde{\mathcal{S}}^0(U)$  and  $\rho \in U$ , observe

$$p - p(\rho) \cdot 1 \in \mathcal{M}_\rho$$

implies

$$g(p) - p(\rho) \cdot 1 \in \mathcal{M}_{\kappa(\rho)}.$$

Thus

$$g(p) (\kappa(\rho)) = p(\rho)$$

for every  $\rho \in U$  implies

$$g(p) = p \circ \kappa^{-1}.$$

For each  $\rho \in U$ , let  $(x, \xi)$  be local coordinates for  $X$  in a neighbourhood of  $\rho$  which does not meet  $\partial U$ . Choosing a suitable cutoff  $\chi_\rho$  equal to 1 near  $\rho$ , the  $\chi_\rho x_j$  and  $\chi_\rho \xi_k$  are *approximate coordinates* near  $\rho$ :

$$\begin{aligned} \chi_\rho x_j, \chi_\rho \xi_k &\in \mathcal{S}^0(U) \text{ for all } j, k; \\ \chi_\rho x_j &= x_j, \chi_\rho \xi_k = \xi_k \text{ near } \rho. \end{aligned}$$

Thus

$$(\chi_\rho x_j) \circ \kappa^{-1} \in \mathcal{S}^0(V),$$

and similarly for  $\chi_\rho \xi_j$  for all  $j$ . Composing with inverse coordinate functions in a neighbourhood of  $\kappa(\rho)$  implies  $\kappa^{-1}$  is smooth on  $U$ . The same argument applied to  $g^{-1}$  shows  $\kappa$  is smooth on  $V$ , hence a diffeomorphism.  $\square$

**Lemma 3.3.** *The diffeomorphism  $\kappa$  constructed in Lemma 3.2 is symplectic.*

*Proof.* Observe  $\Psi^0/\Psi^{-\infty}(U)$  is a Lie algebra with brackets  $ih^{-1}[\cdot, \cdot]$ , and  $g$  induces a Lie algebra isomorphism with  $\Psi^0/\Psi^{-\infty}(V)$ .  $\mathcal{S}^0(U)$  is a Lie algebra with brackets  $\{\cdot, \cdot\}$ , hence  $g_0$  is a Lie algebra isomorphism  $\mathcal{S}^0/\mathcal{S}^{-1}(U) \rightarrow \mathcal{S}^0/\mathcal{S}^{-1}(V)$ . Let  $a, b \in \mathcal{S}^0(U)$  and calculate

$$g_0(\{a, b\}) = \{g_0(a), g_0(b)\},$$

or

$$(\{a, b\}) \circ \kappa^{-1} = \{a \circ \kappa^{-1}, b \circ \kappa^{-1}\}.$$

Letting  $a$  and  $b$  run through local approximate coordinates implies  $\kappa^{-1}$  is symplectic.  $\square$

Now fix  $\tilde{U} \Subset U$ , and let

$$F : L^2(X) \rightarrow L^2(Y)$$

be the  $h$ -Fourier integral operator associated to  $\kappa|_{\tilde{U}}$  as in Proposition 2.3. We define an automorphism of  $\Psi^0/\Psi^{-\infty}(\tilde{U})$ ,  $g_1$ , by

$$(3.3) \quad g_1(P) = F^{-1}g(P)F.$$

Observe  $g_1$  is both order-preserving and preserves principal symbol.

**Lemma 3.4.** *Suppose*

$$g_1 : \Psi^0/\Psi^{-\infty}(\tilde{U}) \rightarrow \Psi^0/\Psi^{-\infty}(\tilde{U})$$

*is an order-preserving automorphism which preserves principal symbol. Then there exists  $B \in \Psi^0(\tilde{U})$ , elliptic on  $\tilde{U}$  such that*

$$(3.4) \quad g_1(P) = BPB^{-1} \bmod \mathcal{O}(h^\infty)$$

*for every  $P \in \Psi^0/\Psi^{-\infty}(U)$ .*

*Proof.* The proof will be by induction. We drop the dependence on  $\tilde{U}$  since the lemma is concerned with automorphisms. Suppose for  $l \geq 1$  we have for every  $m$  and every  $P \in \Psi^m$

$$g_1(P) - P \in \Psi^{m-l}.$$

This induces a map

$$\beta : \mathcal{S}^m/\mathcal{S}^{m-1} \rightarrow \mathcal{S}^{m-l}/\mathcal{S}^{m-l-1},$$

which, using the Weyl composition formula (2.1), satisfies

- (i)  $\beta(pq) = \beta(p)q + p\beta(q)$ ;
- (ii)  $\beta(\{p, q\}) = \{\beta(p), q\} + \{p, \beta(q)\}$ .

Consider the action of  $\beta$  on  $\mathcal{S}^0$ , and observe from property (i) above, for  $p, q \in \mathcal{S}^0$ ,

$$\beta(pq) = \beta(p)q + p\beta(q) \in \mathcal{S}^{-l},$$



so  $\beta$  is  $h^l$  times a derivation on  $\mathcal{S}^0$ .

For any  $\rho \in U$ , we choose coordinates  $(x, \xi)$  near  $\rho$ , and a cutoff  $\chi_\rho$  which is equal to 1 near  $\rho$  and compactly supported in  $U$ . Then  $\chi_\rho x_j$  and  $\chi_\rho \xi_j$  become approximate coordinates which are equal to  $x_j$  and  $\xi_j$  near  $\rho$  but are in  $\mathcal{S}^0$ . Near  $\rho$ ,  $\beta$  takes the form

$$\beta = h^l \sum_j (\gamma_j(x, \xi) \partial_{x_j} + \delta_j(x, \xi) \partial_{\xi_j}),$$

where  $\gamma_j = \beta(\chi_\rho x_j)$  and  $\delta_j = \beta(\chi_\rho \xi_j)$ . Using property (ii) above, we have near  $\rho$

$$\beta(\{\chi_\rho x_j, \chi_\rho \xi_k\}) = \beta(\{\chi_\rho x_j, \chi_\rho x_k\}) = \beta(\{\chi_\rho \xi_j, \chi_\rho \xi_k\}) = 0$$

which implies

$$\frac{\partial \gamma_j}{\partial x_k} = -\frac{\partial \delta_k}{\partial \xi_j}, \quad \frac{\partial \gamma_j}{\partial x_k} = \frac{\partial \gamma_k}{\partial x_j}, \quad \text{and} \quad \frac{\partial \delta_j}{\partial \xi_k} = \frac{\partial \delta_k}{\partial \xi_j}.$$

Thus there exists a locally defined smooth function  $f$  such that

$$\gamma_j = \frac{\partial f}{\partial \xi_j} \quad \text{and} \quad \delta_k = -\frac{\partial f}{\partial x_k},$$

and locally

$$h^{-l} \beta = H_f.$$

Define a smooth function  $b$  by

$$b = \exp(-i\chi f),$$

for a cutoff  $\chi$  which is identically 1 on  $\tilde{U}$  with support in  $U$ , so that  $df = idb/b$  on  $\tilde{U}$ , and locally

$$\beta = h^l H_{i \log b}.$$

Let  $B = \text{Op}_h^w(b)$  and observe the principal symbol of

$$B^{-1}PB - P = B^{-1}[P, B]$$

in the  $\mathcal{S}^0(\tilde{U})$  calculus is

$$\frac{h}{i} b^{-1} \{p, b\} = h H_{i \log b}(p).$$

For the base case of our induction, if  $P \in \mathcal{S}^m(\tilde{U})$ , then

$$g_1(P) - B^{-1}PB \in \mathcal{S}^{m-2}(\tilde{U}),$$

so that

$$Bg_1(P)B^{-1} - P \in \mathcal{S}^{m-2}(\tilde{U}).$$

Replace  $g_1(P)$  with  $Bg_1B^{-1}$ .

Now for the purposes of induction, assume

$$g_1(P) - P \in \mathcal{S}^{m-l}(\tilde{U}),$$

and apply the above argument to get  $B_l \in \mathcal{S}^{-l}$  so that

$$g_1(P) - B_l^{-1}PB_l \in \mathcal{S}^{m-l-1}(\tilde{U}).$$

Then replacing  $g_1(P)$  with  $B_l g_1(P) B_l^{-1}$  finishes the induction. Thus there exists  $B \in \mathcal{S}^0(\tilde{U})$  so that

$$B g_1(P) B^{-1} = P \bmod \mathcal{O}(h^\infty).$$

□

Theorem 1 now follows immediately from applying Proposition 2.1 to (3.3) and (3.4).

□

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